A Procedure for Defining Behavior of Weight Functions Near the Edge for Best Convergence Using the Galerkin Method

Pinkhos Sh. Fridberg and Iosif M. Yakover

***Abstract*—A general procedure is described for determining the behavior of the weight functions (WF) near the edge so as to provide the best convergence using the Galerkin Method (GM) for calculating linear functionals of the electromagnetic theory. It is believed that our procedure is proposed for the first time. The procedure is based on the equivalence of two methods of calculating such functionals—the GM and the variational method (VM). To implement the procedure the sought linear functional is expressed as a variational functional. The stationarity condition of the latter leads to some auxiliary problem. Due to the equivalence mentioned above the WF behavior near the edge is the same as that obtained from the solution of the auxiliary problem. The efficiency of the procedure i.e. high speed of convergence is illustrated by two examples: 1) calculation of the equivalent circuit shunt impedance of the capacitive diaphragm in a plane waveguide, and 2) calculation of the capacitance of a metal tube segment filled with dielectric.**

I. Introduction

The MAJORITY of problems in electromagnetic theory reduce to the calculation of the linear functional F:

$$F = (u, \tilde{f}) \equiv \int_{\alpha}^{1} dx u(x) \tilde{f}(x), \qquad -1 \leq a < 1, \quad (1)$$

where $u(x)$ is the solution of the linear integral equation (IE)

$$Lu \equiv \int_{a}^{1} dx' K(x, x') u(x') + qu(x) = f(x), \qquad x \in [a, 1] \quad (2)$$

of the corresponding boundary problem, and $\tilde{f}$ is the given function.

The functional F is usually calculated by the GM [1]. The solution is expanded into a series:

$$u(x) = \sum_{\mu=1}^{\infty} A_\mu \psi_\mu(x), \quad (3)$$

where $\{\psi_\mu\}$ is some complete system of basis functions (BF). Afterwards IE (2) is averaged with functions $\tilde{\psi}_\nu$, $\nu = 1, 2, \cdots$, comprising a complete system $\{\tilde{\psi}_\nu\}$ of weight functions (WF). This results in the infinite system of linear algebraic equations (SLAE)

$$\sum_{\mu=1}^{\infty} A_\mu (\tilde{\psi}_\nu, L\psi_\mu) = (\tilde{\psi}_\nu, f), \qquad \nu = 1, 2, \cdots, \quad (4)$$

which is usually solved by the reduction method. The value of F in the Mth approximation (M is the order of the reduced SLAE) is calculated by

$$F_M = \sum_{\mu=1}^{M} A_\mu^M (\psi_\mu, \tilde{f}). \quad (5)$$

In this method the convergence rate with the increase of M depends substantially on the behavior of BF and WF near the ends of the segment $[a, 1]$. In the physical problem the edges of the investigated object usually correspond to those ends. Thus the question arises: how must BF and WF behave near the edge of ensure the best convergence of the GM? The obvious answer for BF is that they must behave like the solution $u(x)$ of the problem described by the IE (2). Concerning WF this question was never discussed, they were submitted to the same edge conditions as BF.

The present work is aimed at developing of the general procedure for defining the WF behavior near the edge, which ensures the best convergence using the GM.

II. The Equivalence of the GM and VM

To determine the properties of $\tilde{\psi}_\nu$ functions let us use VM and show its equivalence to the GM. Using the IE (2) we present the expression (1) as the variational functional:

$$F\{u, \tilde{u}\} = (u, \tilde{f}) + (\tilde{u}, f) - (\tilde{u}, Lu). \quad (6)$$

This functional reaches the stationary value when u and $\tilde{u}$ are solutions of two problems: 1) the original one described by IE (2), and 2) some auxiliary[1], the IE of which

[1]From now on the values describing the auxiliary problem are marked with tilde.

Manuscript received May 6, 1991; revised January 21, 1992.

P. Sh. Fridberg is with the Zondas Company, P.O. Box 1683, Vilnius, 232010 Lithuania.
I. M. Yakover is with Department of Electrical Engineering-Physical Electronics, Faculty of Engineering, Tel-Aviv University, Ramat Aviv, 69978 Israel.

IEEE Log Number 9200853.

is obtained by the equating to zero of the variation of (6) with respect to u

$$\tilde{L}\tilde{u} \equiv \int_a^1 dx' \tilde{K}(x, x')\tilde{u}(x') + q\tilde{u}(x) = \tilde{f}(x), \quad x \in [a, 1]. \tag{7}$$

The integral operator $\tilde{L}$ of the auxiliary problem is determined as follows:

$$(\tilde{u}, Lu) = (u, \tilde{L}\tilde{u}). \tag{8}$$

Note that F may be also expressed in the terms of the auxiliary problem

$$F = (\tilde{u}, f). \tag{9}$$

Let us expand $\tilde{u}(x)$ into a series with respect to the complete system of $\tilde{\psi}_\nu$ functions

$$\tilde{u}(x) = \sum_{\nu=1}^{\infty} \tilde{A}_\nu \tilde{\psi}_\nu(x) \tag{10}$$

and substitute this expansion as well as (3) into the expression (6). The result is

$$F\{u, \tilde{u}\} = \sum_{\mu=1}^{\infty} A_\mu(\psi_\upsilon, \tilde{f}) + \sum_{\nu=1}^{\infty} \tilde{A}_\nu(\tilde{\psi}_\nu, f) - \sum_{\mu,\nu=1}^{\infty} \tilde{A}_\nu A_\upsilon(\tilde{\psi}_\nu, L\psi_\mu). \tag{11}$$

If we equate derivatives with respect to $\tilde{A}_\nu$ to zero, then we get SLAE (4). This proves the equivalence of the GM to VM. Note that if derivatives of (11) with respect to A_μ are equated to zero then we obtain SLAE of the auxiliary problem

$$\sum_{\nu=1}^{\infty} \tilde{A}_\nu(\psi_\mu, \tilde{L}\tilde{\psi}_\nu) = (\psi_\mu, \tilde{f}), \qquad \mu = 1, 2, \cdots, \tag{12}$$

The F value in the Nth approximation (N is the order of the reduced system (12)) is calculated by

$$F_N = \sum_{\nu=1}^{N} \tilde{A}_\nu^N(\tilde{\psi}_\nu, f). \tag{13}$$

III. Choice of WF and BF Systems

The equivalence of the two methods (the GM and VM) used to calculate linear functionals allows us to conclude that WF of the original problem convert into the BF of the auxiliary problem and vice versa. It means that the best convergence of the GM is achieved when each of the functions of $\tilde{\psi}_\nu(x)$ behaves like $\tilde{u}(x)$ near the ends of the segment $[a, 1]$, while each of the functions of $\psi_\mu(x)$ behaves like $u(x)$.

The behavior of functions u and $\tilde{u}$ near the edge may usually be determined by considering a corresponding two-dimensional model structure. Methods of investigation of electromagnetic fields in such structures were developed in [2]–[5]. Sectorial regions formed by metal and dielectric wedges were considered in [2], and those formed by metal and ferrite wedges in [3]. In [4], [5] the resistive half-plane taken separately as well as in conjunction with the metal wedge was studied.

The behavior of solutions of both original and auxiliary problems may be described as

$$u(x) \sim (x - a)^\alpha, \quad \tilde{u}(x) \sim (x - a)^{\tilde{\alpha}}, \qquad x \to a, \tag{14a}$$

$$u(x) \sim (1 - x)^\beta, \quad \tilde{u}(x) \sim (1 - x)^{\tilde{\beta}}, \qquad x \to 1, \tag{14b}$$

The values of the α, β, $\tilde{\alpha}$, $\tilde{\beta}$, parameters can not be less than some definite values prescribed by a so-called edge condition [6]. This condition requires the electromagnetic energy concentrated near the edge to be finite, thus assuming that uniqueness theorem is fulfilled. Usually polynomials with corresponding weights are used as ψ_μ and $\tilde{\psi}_\nu$ functions with properties (14). Since the convergence is practically independent of the polynomial type, they should be chosen in such a way as to make matrix elements of SLAE (4) as simple and convenient for calculations as possible. The possibility of such choice depends on the type of kernel $K(x, x')$.

In most cases the kernel $K(x, x')$ may be expressed as an integral (or a series) in the pulse space. In such cases there is an opportunity to set the corresponding integral transforms (of BF and WF) rather than the BF and the WF themselves (see Appendix).

Thus the procedure of choice of the WF system ensuring the best convergence of the GM is as follows. IE of the auxiliary problem is formulated using VM. Subsequently the physical meaning of the problem must be established. Next two-dimensional model structures are defined, on which investigation are made to determine the behavior of $\tilde{u}(x)$ and thus of $\tilde{\psi}_\nu(x)$ near the segment ends $[a, 1]$. Then $\tilde{\psi}_\nu$ functions are set either by relations (A1)–(A4) or as other polynomials with corresponding weights. Let us illustrate our procedure using the following numerical examples.

IV. Capacitive Diaphragm in a Plane Waveguide

Consider an infinite plane waveguide with a symmetric infinitesimally thin diaphragm in its cross-section (Fig. 1). Let the incident TEM mode propagate towards the diaphragm from the region $z = -\infty$. The magnetic field[2] of the TEM mode is

$$H_x(y, z) = I \exp(ik_0 z), \tag{15}$$

where k_0 is the wavenumber in free space, I is the dimensional factor. If we limit ourselves to the $0 < k_0 b < 2\pi$ frequency band, then only one propagating mode is excited in the waveguide and the system considered is described by the equivalent circuit. Calculation of the equivalent circuit shunt impedance X

$$X = \int_{-1}^{1} dv u(v), \qquad v = 2y/d \tag{16}$$

[2]Time dependence is presented as $\exp(-i\omega t)$.

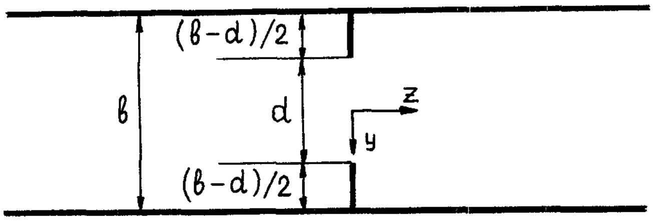

Fig. 1. Capacitive diaphragm in a plane waveguide.

requires the solution of IE

$$\int_{-1}^{1} dv' \eta(v, v') u(v') = 1, \qquad -1 \leq v \leq 1, \quad (17)$$

where

$$\eta(v, v') = 4k_0 \sum_{n=1}^{\infty} \cos(p_n v) \cos(p_n v')/q_n,$$

$q_n = \sqrt{(2\pi n/b)^2 - k_0^2}$, $p_n = \pi dn/b$, $u(v)$ is a non-dimensional function proportional to the tangential component of the electric field at the aperture. Near the edges of the diaphragm the behavior of this function is [7]:

$$u(v) \sim (1 - v^2)^{-1/2}, \; v \to \pm 1. \quad (18)$$

Using the GM we set BF by (A2), replacing μ by $2\mu - 1$, because $u(v)$ is even. Taking into account (18) it is easy to see that $\tau = -1/2$ is the exact value for substitution in (A2). However, the following calculations will be carried out for various admissible values of τ. WF is also set by (A2) relation, assuming $\tau = \tilde{\tau}$ and $\mu = 2\nu - 1$. The result is the SLAE

$$\sum_{\mu=1}^{M} T_{\nu\mu} A_\mu^M = \delta_{1\nu} d_1(\tilde{\tau}), \qquad \nu = 1, 2, \cdots, M, \quad (19)$$

where

$$T_{\nu\mu} = 4k_0 \sum_{n=1}^{\infty} \Phi_{2\nu-1}(\tilde{\tau} | p_n) \Phi_{2\mu-1}(\tau | p_n)/q_n,$$

$$d_1(\tilde{\tau}) = 1/2^{\tilde{\tau}+1/2} \Gamma(\tilde{\tau} + 3/2),$$

$\Gamma(x)$ is gamma function, $\delta_{1\nu}$ is Kronecker delta. The impedance X sought is calculated via:

$$X_M = A_1^M d_1(\tau) \quad (20)$$

resulting from (16).

To determine the parameter $\tilde{\tau}$, providing the best convergence of the GM, let us formulate an auxiliary problem. We present (16) as a variational functional

$$X\{u, \tilde{u}\} = \int_{-1}^{1} dv u(v) + \int_{-1}^{1} dv \tilde{u}(v) - \int_{-1}^{1} dv \int_{-1}^{1} dv' \tilde{u}(v) \eta(v, v') u(v'). \quad (21)$$

By equating the variation of (21) with respect to u to zero, it is evident that IE of the auxiliary problem is equivalent to IE (17) of the original problem. It means $\tilde{u} \equiv u$, i.e. the best convergence of the GM is achieved when BF and WF behave identically near the edge ($\tilde{\tau} = -1/2$). This is the most typical example.

Table I gives the calculated values of X_M providing $k_0 b = 5$, $d/b = 0.5$. They show the convergence dependence of the GM and τ and $\tilde{\tau}$ parameters. As is seen, the best convergence is really achieved at $\tau = \tilde{\tau} = -1/2$.

V. Dielectric Cylinder with Metal-Coated Side Surface

Now we shall consider the problem where the best convergence of the GM is achieved if BF and WF behave differently near the edges. This kind of problem has not been investigated previously.

To calculate the normalized electric capacity C normalized to $4\pi\epsilon^+ r$ of a metal shell on the side surface of the dielectric cylinder (Fig. 2) we use the expression

$$C = h \int_{-1}^{1} d\zeta u_1(\zeta) + 2 \int_0^1 dx x u_2(x). \quad (22)$$

Here $u_1(\zeta)$ and $u_2(x)$ are non-dimensional functions proportional to the discontinuity of the normal derivative of the electrostatic potential on the side surface of cylinder and on its end face respectively. These functions satisfy the following IE system:

$$\begin{cases} h \int_{-1}^{1} d\zeta' R_{11}(\zeta, \zeta') u_1(\zeta') + 2 \int_0^1 dx x R_{12}(\zeta, x) u_2(x) = 1 \\ \lambda h \int_{-1}^{1} d\zeta R_{21}(x, \zeta) u_1(\zeta) + u_2(x) + \lambda \int_0^1 dx' x' R(x, x') u_2(x') = 0, \end{cases} \quad (23)$$

$$R_{11}(\zeta, \zeta') = \int_0^\infty dk f_{11}(k) \cos(kh\zeta) \cos(kh\zeta'),$$

$$f_{11}(k) = (2/\pi) K_0(k) I_0(k),$$

$$R_{12}(\zeta, x) = \int_0^\infty dk f_{12}(k) \cos(kh\zeta) I_0(kx),$$

$$f_{12}(k) = (2/\pi) K_0(k) \cos(kh),$$

$$R_{21}(x, \zeta) = \int_0^\infty dk f_{21}(k) I_0(kx) \cos(kh\zeta),$$

$$f_{21}(k) = -(2\pi) k K_0(k) \sin(kh),$$

$$R(x, x') = \int_0^\infty dk f(k) J_0(kx) J_0(kx'),$$

$$f(k) = -k \exp(-2kh),$$

$\zeta = z/l$, $x = \rho/r$, $h = l/r$, $\lambda = (\epsilon^- - \epsilon^+)/(\epsilon^- + \epsilon^+)$, ϵ^- and ϵ^+ are the dielectric permittivities of the cylinder and its environment, J_μ is the Bessel function, I_μ and K_μ are modified Bessel functions. Writing (22) and (23) we

TABLE I
DEPENDENCE OF CONVERGENCE SPEED OF VALUE X_M ON PARAMETERS $\tilde{\tau}$ AND τ

		τ				
$\tilde{\tau}$	M	−0, 9	−0, 7	−0, 5	−0, 3	−0, 1
	1	−5.9456	−2.4665	−1.4250	−0.9439	−0.6715
	2	−1.8703	−1.6933	−1.5763	−1.4925	−1.4293
−0, 9	3	−1.6483	−1.6081	−1.5750	−1.5474	−1.5240
	4	−1.6087	−1.5908	−1.5750	−1.5612	−1.5488
	5	−1.5943	−1.5842	−1.5750	−1.5667	−1.5644
	1	−2.4665	−1.8020	−1.5293	−1.3881	−1.3041
	2	−1.6933	−1.6243	−1.5757	−1.5393	−1.5109
−0, 7	3	−1.6081	−1.5901	−1.5751	−1.5623	−1.5514
	4	−1.5908	−1.5824	−1.5751	−1.5685	−1.5626
	5	−1.5842	−1.5794	−1.5751	−1.5711	−1.5674
	1	−1.4250	−1.5293	−1.5927	−1.6355	−1.6664
	2	−1.5763	−1.5757	−1.5751	−1.5744	−1.5738
−0, 5	3	−1.5750	−1.5751	−1.5751	−1.5750	−1.5750
	4	−1.5750	−1.5751	−1.5751	−1.5750	−1.5750
	5	−1.5750	−1.5751	−1.5751	−1.5750	−1.5750
	1	−0.9439	−1.3881	−1.6355	−1.7920	−1.8998
	2	−1.4925	−1.5393	−1.5744	−1.6018	−1.6237
−0, 3	3	−1.5474	−1.5623	−1.5750	−1.5860	−1.5956
	4	−1.5612	−1.5685	−1.5750	−1.5809	−1 5862
	5	−1.5667	−1.5711	−1.5750	−1.5787	−1.5821
	1	−0.6715	−1.3041	−1.6664	−1.8998	−2.0622
	2	−1.4293	−1.5109	−1.5738	−1.6237	−1.6643
−0, 1	3	−1.5240	−1.5514	−1.5750	−1.5926	−1.6135
	4	−1.5488	−1.5626	−1.5750	−1.5862	−1.5963
	5	−1.5644	−1.5674	−1.5750	−1.5821	−1.5885

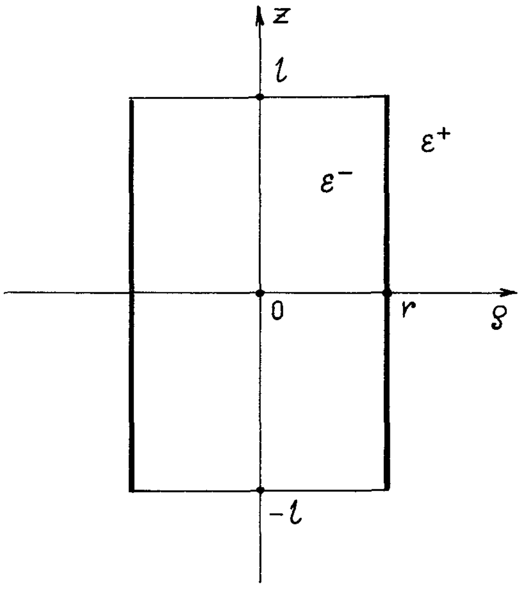

Fig. 2. Dielectric cylinder with metalized side surface.

consider the problem's symmetry with regard to the plane $z = 0$.

The behavior of functions $u_1(\zeta)$ and $u_2(x)$ near the edge can be determined by solving two-dimensional model problem: the rectangular dielectric wedge of which has one side metal-coated. The permittivity of the wedge and its environment is respectively ϵ^- and ϵ^+. Thus [6]

$$u_1(\zeta) \sim (1 - \zeta^2)^{\tau_0}, \qquad \zeta \to \pm 1,$$
$$u_2(x) \sim (1 - x)^{\tau_0}, \qquad x \to 1,$$
$$\tau_0 = -1/2 + (1/\pi) \arcsin(\lambda/2). \tag{24}$$

To implement the GM functions u_1 and u_2 are exapnded in series of the form

$$u_1(\zeta) = \sum_{\mu=1}^{\infty} A_\mu \chi_\mu(\zeta),$$
$$u_2(x) = \sum_{\nu=1}^{\infty} B_\nu \psi_\nu(x). \tag{25}$$

Complete systems $\{\chi_\mu\}$ and $\{\psi_\nu\}$ of BF are set by relations (A2) and (A3). We must take into account that $u_1(\zeta) = u_1(-\zeta)$ and replace μ by $2\mu - 1$. We have carried out calculations, showing how the choice in (A2) and (A3) of the parameter τ influences the speed of convergence of GM. That is why, we do not substitute τ_0 from (24) in (A2) and (A3). We designate complete systems of WF as $\{\tilde{\chi}_\mu\}$ and $\{\tilde{\psi}_\nu\}$. Set WF $\tilde{\chi}_\mu$ by (A2) assuming $\tau = \tilde{\tau}_1$, and replacing μ by $2\mu - 1$. Set WF $\tilde{\psi}_\nu$ by (A3) assuming $\tau = \tilde{\tau}_2$. Thus SLAE is

$$\sum_{\mu=1}^{M} R^{11}_{\mu'\mu} A^M_\mu + \sum_{\nu=1}^{N} R^{12}_{\mu'\nu} B^N_\nu = \delta_{1\mu'} \, d_1(\tilde{\tau}_1), \qquad 1 \le \mu' \le M,$$
$$\sum_{\mu=1}^{M} R^{21}_{\nu'\mu} A^M_\mu + \sum_{\nu=1}^{N} (R^{22}_{\nu'\nu} + R_{\nu'\nu}) B^N_\nu = 0, \qquad 1 \le \nu' \le N, \tag{26}$$

$$R_{\mu\nu}^{11} = h \int_0^\infty dkf_{11}(k)\Phi_{2\mu-1}(\tilde{\tau}_1|kh)\Phi_{2\nu-1}(\tau|kh),$$

$$R_{\mu\nu}^{12} = 2 \int_0^\infty dkf_{12}(k)\Phi_{2\mu-1}(\tilde{\tau}_1|kh)I_\nu^0(\tau|k),$$

$$R_{\mu\nu}^{21} = \lambda h \int_0^\infty dkf_{21}(k)I_\mu^0(\tilde{\tau}_2|k)\Phi_{2\nu-1}(\tau|kh),$$

$$R_{\mu\nu}^{22} = \frac{\Gamma(\tau + \tilde{\tau}_2 + 1)\Gamma(\mu + \nu - 1)}{2^{\tau+\tilde{\tau}_2+1}\Gamma(\mu - \nu + \tilde{\tau}_2 + 1)\Gamma(\nu - \mu + \tau + 1)\Gamma(\nu + \mu + \tau + \tilde{\tau}_2)},$$

$$R_{\mu\nu} = \lambda \int_0^\infty dkf(k)J_\mu^0(\tilde{\tau}_2|k)J_\nu^0(\tau|k),$$

$$d(\tilde{\tau}_1) = 1/2^{\tilde{\tau}_1+1/2}\Gamma(\tilde{\tau}_1 + 3/2).$$

The normalized capacitance C sought is calculated by

$$C_{MN} = [hA_1^M - \sqrt{2}\,\Gamma(\tau + 3/2)B_1^N/\Gamma(\tau + 2)]\, d_1(\tau), \tag{27}$$

resulting from (22).

In order to determine parameters $\tilde{\tau}_1$ and $\tilde{\tau}_2$ ensuring the best convergence of the GM let's formulate the corresponding auxiliary problem. Let's present (22) in the form of the variational functional

$$\begin{aligned} C\{u_1, u_2; \tilde{u}_1, \tilde{u}_2\} &= h \int_{-1}^1 d\zeta u_1(\zeta) + 2 \int_0^1 dx x u_2(x) + \int_{-1}^1 d\zeta \tilde{u}_1(\zeta) \\ &- h \int_{-1}^1 d\zeta \int_{-1}^1 d\zeta' \tilde{u}_1(\zeta) R_{11}(\zeta, \zeta') u_1(\zeta') - 2 \int_{-1}^1 d\zeta \int_0^1 dx \tilde{u}_1(\zeta) R_{12}(\zeta, x) x u_2(x) \\ &- \lambda h \int_0^1 dx \int_{-1}^1 d\zeta \tilde{u}_2(x) x R_{21}(x, \zeta) u_1(\zeta) - \int_{-1}^1 dx x \tilde{u}_2(x) u_2(x) \\ &- \lambda \int_0^1 dx \int_0^1 dx' \tilde{u}_2(x) \cdot x R_{22}(x, x') x' u_2(x'). \end{aligned} \tag{28}$$

Equating to zero of variations of this functional with respect to u_1 and u_2 we obtain the IE system

$$\begin{aligned} &\int_{-1}^1 d\zeta' R_{11}(\zeta, \zeta')\tilde{u}_1(\zeta') + \lambda \int_0^1 dx x R_{21}(x, \zeta)\tilde{u}_2(x) = 1 \\ &\int_{-1}^1 d\zeta R_{12}(\zeta, x)\tilde{u}_1(\zeta) + \tilde{u}_2(x)/2 + (\lambda/2) \int_0^1 dx' x' R_{22}(x, x')\tilde{u}_2(x') = 1, \end{aligned} \tag{29}$$

where $\tilde{u}_1$ and $\tilde{u}_2$ functions differ by constant factors from the solutions of the IE system described in the following problem. Charged to a unit potential the metal shell covers the side surface of a cylinder with permittivity ϵ^+. The cylinder's environment has permittivity ϵ^- and charges are distributed on inner and outer sides of the end faces of the cylinder with such density as to satisfy the boundary conditions:

$$\begin{aligned} &\epsilon^- \tilde{\phi}|_{\zeta=1+0} - \epsilon^+ \tilde{\phi}|_{\zeta=1-0} = \epsilon^- - \epsilon^+, \\ &\left.\frac{\partial\tilde{\phi}(\zeta)}{\partial\zeta}\right|_{\zeta=1+0} = \left.\frac{\tilde{\partial}\phi(\zeta)}{\partial\zeta}\right|_{\zeta=1-0}, \\ &\epsilon^- \tilde{\phi}|_{\zeta=-1+0} - \epsilon^+ \tilde{\phi}|_{\zeta=-1-0} = \epsilon^- - \epsilon^+, \\ &\left.\frac{\partial\tilde{\phi}(\zeta)}{\partial\zeta}\right|_{\zeta=-1+0} = \left.\frac{\partial\tilde{\phi}(\zeta)}{\partial\zeta}\right|_{\zeta=-1-0}, \end{aligned} \tag{30}$$

where $\tilde{\phi}$ is the electrostatic potential. The function $\tilde{u}_1$ proves to be proportional to the discontinuity of a normal (to the side surface) derivative of the potential $\tilde{\phi}$, while function $\tilde{u}_2$ proves to be proportional to the discontinuity $\tilde{\phi}$ on the end face of the cylinder. It should be noted that the original and auxiliary problems differ because the kernel of the system (23) is non-symmetrical.

Let's now determine the behavior of the solution of the auxiliary problem near the edge. For that purpose we consider a two-dimensional model structure like a rectangular dielectric wedge with permittivity ϵ^+. This wedge has a metal-coated face and is placed into the environment with the dielectric side permittivity ϵ^-. Charges are distributed on both inner and outer sides of the metal-free face with such a density that both the tangential component of the electric induction vector and the normal component of the electric field vector are continuous. Analysis of this structure shows that $\tilde{u}_1$ and $\tilde{u}_2$ function behave at the edge as

TABLE II
DEPENDENCE OF CONVERGENCE SPEED OF VALUE C_{MM} ON PARAMETERS $\tilde{\tau}_1 = \tilde{\tau}_2 - 1$ AND τ

		τ				
$\tilde{\tau}_1 = \tilde{\tau}_2 - 1$	M	$-2/3$	-0.5	$\tau_0 = -0\ 36585$	$-1/3$	0
	1	1.1434	1.1702	1.1841	1.1868	1 2059
	2	1.1802	1.1831	1.1851	1.1856	1.1897
$-2/3$	3	1.1836	1.1845	1.1851	1.1853	1 1868
	4	1.1845	1.1848	1.1852	1.1852	1.1860
	5	1.1848	1.1850	1.1852	1.1852	1.1856
	1	1.1814	1.1841	1.1851	1 1853	1 1861
	2	1.1837	1.1846	1.1851	1.1853	1.1863
$-0,5$	3	1.1846	1.1849	1.1852	1.1852	1.1857
	4	1.1849	1.1850	1.1852	1.1852	1.1854
	5	1.1850	1.1851	1.1852	1.1852	1 1853
	1	1.2021	1.1918	1.1857	1.1844	1.1747
	2	1.1862	1.1856	1.1852	1.1850	1.1840
$\tau_0 = -0.36585$	3	1.1854	1.1853	1.1852	1.1851	1.1848
	4	1.1852	1.1852	1.1852	1.1851	1.1850
	5	1.1852	1.1852	1.1852	1.1851	1.1851
	1	1.2062	1.1933	1.1858	1.1842	1.1724
	2	1.1868	1.1859	1.1852	1.1850	1.1834
$-1/3$	3	1.1856	1.1853	1.1852	1.1851	1.1846
	4	1.1853	1.1852	1.1852	1.1851	1.1849
	5	1.1852	1.1852	1.1852	1.1851	1.1850
	1	1.2364	1.2048	1.1867	1.1829	1.1546
	2	1.1917	1.1880	1.1852	1.1846	1.1768
0	3	1.1873	1.1861	1.1852	1 1849	1.1827
	4	1.1861	1.1856	1.1852	1 1850	1.1839
	5	1.1857	1.1854	1.1852	1.1851	1.1844

follows:

$$\tilde{u}_1(\zeta) \sim (1 - \zeta^2)^{\tau_0}, \qquad \zeta \to \pm 1,$$
$$\tilde{u}_1(x) \sim (1 - x)^{1+\tau_0}, \qquad x \to 1. \tag{31}$$

It is evident that the behavior near the edge of the auxiliary problem solution differs substantially from that of the original one. This fact has not been discovered prior to this publication.

Values of C calculated by solving SLAE (26), provide that $h = 1$, $\epsilon^- = 10\epsilon^+$, $M = N$ and for different values of parameters τ and $\tilde{\tau}_1 = \tilde{\tau}_2 - 1$, are shown in Table II. Obviously the best convergence of the GM is obtained when $\tau = \tilde{\tau}_1 = \tilde{\tau}_2 - 1 = \tau_0$, which completely agrees with the above determined behavior of auxiliary problem solutions (31) near the edge.

VI. CONCLUSION

The procedure for choice of the weight functions system ensuring the best convergence of the GM was developed. To implement this procedure the linear functional sought is expressed as a variational functional, the stationarity condition of which reduces to some auxiliary problem. Proceeding from the physical meaning of the latter the corresponding two-dimensional model structure is considered. Its analysis allows to determine the behavior of the auxiliary problem solution near the edge. The equivalence of two methods of linear functionals calculation—the GM and VM—signifies that both weight (basis) functions and the auxiliary (original) problem solution must behave identically near the edge.

The efficiency of the suggested procedure is illustrated by two numerical examples: calculation of the equivalent circuit shunt impedance of a capacitive diaphragm inserted into a plane waveguide, and the calculation of the capacitance of a metal shell covering the side surface of a dielectric cylinder. In the first problem the best convergence of the GM is achieved when BF and WF behave identically near the edge while in the second problem the desired result is obtained when their behavior differs. The latter fact was discovered for the first time.

APPENDIX

If the kernel $K(x, x')$ is expressed in the form of Fourier integral (series) and $a = -1$, then the complete system of functions ψ_μ satisfying (14) is set in accordance with [8] by the relation[3]:

$$\Phi_\mu(\alpha, \beta | p) \equiv \int_{-1}^{1} dx \exp(-ipx) \psi_\mu(x) = p^{\mu-1} \exp(ip)\, {}_1F_1(\mu + \alpha, 2\mu + \alpha + \beta; -2ip), \qquad \mu = 1, 2, \cdots, \tag{A1}$$

where ${}_1F_1$ is the degenerate hypergeometric Cummer's series [10]. That corresponds to the use of Jacobi polynomials with proper weights for the ψ_μ. In a particular case

[3]Later the same result was obtained in [9].

when $\alpha = \beta \equiv \tau > -1$ we have [8]:

$$\Phi_\mu(\tau | p) \equiv \int_{-1}^{1} dx \exp(-ipx)\psi_\mu(x) = p^{-(\tau+1/2)} J_{\mu+\tau-1/2}(p), \quad \mu = 1, 2, \cdots, \tag{A2}$$

instead of (A1). That corresponds to the use of Gegenbauer's polynomials with proper weights for the ψ_μ.

If the kernel $K(x, x')$ is expressed in the form of the integral (series) with respect to Bessel functions with a fixed index s, and $a = 0$, $\alpha = 0$, $\beta \equiv \tau > -1$, then the system of functions $\psi_\mu^s(x)$ is given by relation [11]:

$$\left.\begin{matrix} J_\mu^s(\tau | p) \\ I_\mu^s(\tau | p) \end{matrix}\right\} \equiv \int_0^1 dxx \begin{Bmatrix} J_s(px) \\ I_s(px) \end{Bmatrix} \psi_\mu^s(x) = p^{-(\tau+1)} \begin{cases} J_{2(\mu-1)+s+\tau+1}(p), \\ (-1)^\mu I_{2(\mu-1)+s+\tau+1}(p), \end{cases} \quad \mu = 1, 2, \cdots, \tag{A3}$$

which corresponds to the use of Jacobi polynomials with proper weight for the $\psi_\mu^s(x)$. If the kernel $K(x, x')$ is expanded in series (integral) with respect to arbitrary cylindrical functions with a fixed index s and $a > 0$, $\alpha = \beta \equiv \tau > -1$, then functions ψ_μ^s are set by relations [12], [13]:

$$Z_\mu^s(\tau | p) \equiv \int_a^1 dxxZ_s(px)\psi_\mu^s(x) = p^{-(\tau+1/2)} J_{n(\mu)+\tau+1/2}\left(\frac{1-a}{2}p\right) Z_{m(\mu)+s}\left(\frac{1+a}{2}p\right),$$
$$n(\mu) = [3\mu - 2 - (-1)^\mu(\mu - 2)]/4, \quad m(\mu) = [1 + (-1)^\mu]\mu/4, \quad \mu = 1, 2, \cdots, \tag{A4}$$

where Z_s is an arbitrary linear combination of Bessel functions J_s and Neuman functions N_s, or by the relation:

$$U_\mu^s(\tau | p) \equiv \int_a^1 dxxU_s(px)\psi_\mu^s(x) = (-1)^{\sigma(\mu)} p^{-(\tau+1/2)} \cdot I_{n(\mu)+\tau+1/2}\left(\frac{1-a}{2}p\right) \cdot U_{m(\mu)+s}\left(\frac{1+a}{2}p\right) \tag{A5}$$

where $\sigma = g^+$, if $U_s = I_s$ and $\sigma = g^-$, if $U_s = K_s$; $g^\pm(\mu) = [n(\mu) \pm m(\mu)]/2$. Formulas (A4), (A5) correspond to the use of special functions proposed and investigated by Rosenblum and Fridberg [12], [13] for ψ_μ^s.

References

[1] G. A. Korn, T. M. Korn, *Mathematical Handbook*. New York: McGraw-Hill, 1968.

[2] I. Meixner, "The behavior of electromagnetic fields at edges," *IEEE Trans. Antennas Propagat.*, vol. AP-20, no. 4, pp. 442–446, July 1972.

[3] R. A. Hurd, "On the possibility of intrinsic loss occurring at the edges of ferrites," *Canad. J. Phys.*, vol. 40, pp. 1067–1076, 1962.

[4] I. M. Braver, P. Sh. Fridberg, Kh. L. Garb, and I. M. Yakover, "The behavior of the electromagnetic field near the edge of a resistive half-plane," *IEEE Trans. Antennas Propagat.*, vol. AP-36, no. 12, pp. 1760–1768, Dec. 1988.

[5] I. M. Braver, P. Sh. Fridberg, Kh. L. Garb, and I. M. Yakover, "Electromagnetic field near the common edge of a perfectly conducting wedge and a resistive half-plane," in *Proceedings URSI Int. Symp. of Electromagnetic Theory*, Stokholm, 1989, pp. 88–90.

[6] R. Mittra and S. W. Lee, *Analytical Techniques in the Theory of Guided Waves*. New York: Macmillan, 1971.

[7] J. Schwinger and D. S. Saxon, *Discontinuities in Waveguides*. New York: Gordon and Breach, 1968.

[8] R. F. Fikhmanas and P. Sh. Fridberg, "Use of analytic properties of the Fourier transform in computer realization of variational principles," *Radio Eng. Electron. Phys.*, vol. 23, no. 7, pp. 95–104, July 1978.

[9] L. Mook-Seng and K. P. S. Chandra, "A new class of basic functions for the solution of the E-plane waveguide discontinuity problem," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, no. 8, pp. 705–709, Aug. 1987.

[10] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, vol. 1. New York: McGraw-Hill, 1953.

[11] R. F. Fikhmanas and P. Sh. Fridberg, "Analytical properties of Hankel transform and their use in numerical realization of variational principles," *Radio Eng. Electron. Phys.*, vol. 23, no. 8, pp. 49–54, Aug. 1978.

[12] A. V. Rosenblum and P. Sh. Fridberg, "Integral characteristics of annular inhomogeneities in axially symmetric electrodynamics problems," *Sov. Phys. Doklady*, vol. 30, no. 8, pp. 721–723, Aug. 1985.

[13] A. V. Rosenblum and P. Sh. Fridberg, "Calculation of the linear functionals of one class axially symmetrical problems of electrodynamics," *Sov. J. Commun. Technol. Electron.*, vol. 31, no. 6, pp. 1986.

Pinkhos Sh. Fridberg was born in Vilno, Poland on May 24, 1938. He graduated from Vilnius University in 1961 and received the Ph.D. degree in theoretical and mathematical physics in 1965 and the D.Sci. degree in radio physics in 1974.

At his initiative, the manuscript of Shwinger's notes dealing with discontinuities in waveguides was brought from the United States in 1969 and published in Russian under his editorship in March 1970. From 1961 to 1978 he was Head of the Laboratory of Theoretical Investigation at the Vilnius Scientific Research Institute of Radio Measuring Devices. In 1978 he joined Grodno State University, where he was named Professor. Since 1989 he has been Head of the Department of Theoretical Radiophysics at the Zondas Company, Vilnius.

Dr. Fridberg has published 95 works on applied electrodynamics in leading American and Soviet scientific journals and is the holder of 12 patents.

Iosif M. Yakover was born in Moscow, U.S.S.R., on December 21, 1950. He graduated from Vilnius University, Vilnius, U.S.S.R. in 1973 and received the Ph.D. degree in radio physics from Byelorussian State University, Minsk, U.S.S.R. in 1984.

From 1973 to 1989 he was a Scientific Worker in the Laboratory of Theoretical Investigations at the Vilnius Scientific Research Institute of Radio Measuring Devices. In 1990 he was a Chief Scientist with the Zondas Company. In 1991 he joined Department of Electrical Engineering and Physical Electronics, Tel Aviv University, Tel Aviv, Israel, where he is currently working as a Senior Scientist. His activities include electromagnetic wave scattering by discontinuities in waveguides, analysis of microwave components for millimeter wave applications and three-dimensional electrostatic problem.